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Irregular Sampling Theorems and Series Expansions of Band-Limited Functions

HANS G. FEICHTINGER

Institut für Mathematik, Universität Wien, Strudlhofgasse, 4, A-1090 Wien, Austria

AND

KARLHEINZ GRÖCHENIG*

Department of Mathematics U-9, University of Connecticut, Storrs, Connecticut 06269-3009

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We present a new approach to the problem of irregular sampling of band-limited functions that is based on the approximation and factorization of convolution operators. A special case of our main result is the following theorem: If $\Omega \subseteq R^n$ is compact, $g \in L^1_\alpha(R^n)$ a band-limited function with $\hat{g}(t) \neq 0$ on Ω and $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are two discrete sets in R^n which are "dense enough," then every band-limited function $f \in L^p_w$, $1 \leqslant p < \infty$, has a representation $f = \sum_{j \in J} c_j L_{y_j} g$ as a norm-convergent series, where the coefficients c_j are in l^p_w and can be calculated from the sampled values $f(x_i)$ of f alone. This is a far-reaching generalization of the classical Shannon–Whittacker sampling theorem and can be interpreted as an interpolation method of scattered data by band-limited functions.

Our methods work (a) for a very general class of function spaces, not only weighted L^p_w-spaces, (b) in all dimensions, even in general locally compact abelian groups, (c) to provide the correct behavior of the coefficients in contrast to the traditional estimates currently used by engineers. Furthermore the method is constructive and stable with respect to input errors such as round-off errors, truncation effors, jitter errors, or aliasing errors.

1. INTRODUCTION

The irregular sampling problem asks under which conditions and how a band-limited function can be reconstructed if it is known only at a discrete

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set of points. In the case where the $x_n = \alpha n$ are equally spaced the answer is the classical Shannon–Whittacker–Kotel'nikov sampling theorem. It has been studied extensively along with many variations and has been the subject of a considerable amount of literature, cf. the reviews [6, 25, 24, 34].

Although there is an obvious interest in it, irregular sampling has been investigated in less detail. For a review of the available engineering literature refer to [32], for further motivational remarks refer to [17].

The first studies of this problem in the engineering literature deal with special cases of sampling sets, such as perturbations of the regular sampling at a finite number of points or periodic sets [44], or they are with the context of the jitter error, e.g., [35]. Treating the jitter error gives only an approximation of the original function, but no complete reconstruction, and the estimates are in rather weak norms, usually in the L^∞-norm.

Nevertheless there are many statements that "error-free recovery of signals from irregularly spaced samples" is possible. As an example we mention the so-called "folk theorem": if the sampling rate of $(x_n)_{n \in N}$, $\dots < x_{n-1} < x_n < x_{n+1} < \dots$ is less than the Nyquist rate $1/W$, i.e. $\sup_{n \in N} x_n - x_{n-1} < 1/W$, then there exist band-limited functions $e_{iN} \in L^2(R)$ such that

$$\lim_{N \to \infty} \left\| f - \sum_{i=-N}^{N} f(x_i) e_{iN} \right\|_2 = 0 \tag{1}$$

for every band-limited function $f \in L^2$ with $\operatorname{supp} f \subseteq [-\pi W, \pi W]$, e.g., [3]. Such statements are consequences of gap and density theorems on Fourier series [30]. However, the nature of the functions e_{iN} is unknown; it is also not clear that f has a proper series expansion $f = \sum_i f(x_i) e_i$ with a fixed set of functions e_i.

The mathematical literature offers powerful uniqueness theorems for band-limited functions [1, 2, 28, 29]. With a heavy machinery of complex function theorey general conditions on the sampling set x_i, $i \in I$ are derived, under which a band-limited function f is uniquely determined by the values $f(x_i)$. For a consequence of non-uniqueness see also [41]. The question of how to reconstruct f from the $f(x_i)$ is not touched. This may be the reason why these results are hardly ever referred to in the engineering literature.

For practical purposes a constructive method to recover a band-limited function from its samples is required. All methods we know of are based on the perturbation of the regular sampling theorem. In their original formulation they are mainly results about non-harmonic Fourier series, see [10, 27, 45] and recently [26]. Rephrasings of these results as sampling theorems can be found in [42, 43, 24] among others.

Kadec's $\frac{1}{4}$-theorem provides a reconstruction of a band-limited function $f \in L^2(R)$ with spectrum in $[-\pi, \pi]$ from the sampling values t_n, provided that $\sup_n |t_n - n| < \frac{1}{4}$. In this case the basis functions e_n in the represenation $f(x) = \sum_n f(x_n) e_n$ are "explicitely" known as Lagrange interpolation functions. The e_n's are infinite products of linear terms [44, 23, 43, 39]; they have been of theoretical interest only and have not proved useful in numerical computations. On the contrary, these expanding functions behave unstably and quite unpredictably with respect to a change of the sampling points. It has been shown that a slight modification of only one point may change *all* e_i's dramatically [40, 43].

Duffin and Shaeffer [10] require the condition $|t_n - dn| \leqslant D < \infty$, $\forall n \in Z$ on the sampling set for some fixed numbers $0 < d < 1$ and D. Whereas Kadec's result is equivalent to saying that the functions $e^{it_n\omega}$, $n \in Z$ are an unconditional basis for $L^2(R)$, in [10] the exponentials $e^{it_n\omega}$ constitute only a frame for $L^2(R)$, which is a considerably weaker, but more flexible notion than a basis, see also [45].

Since in the applications only a finite number of (irregular) sampling values is available, "bunched sampling" [34, 44] is often an adequate method for the reconstruction.

Multivariate sampling, which is important in image processing, acoustics, or geophysics, has received considerable attention in the last three years, and several methods are emerging for irregular sampling of images. Duffin-Schaeffer's result has been extended to two dimensions under the special assumption that the sampling set has a product structure [5], see also [33]. Image restoration from a finite number of irregularly spaced samples is treated in [38].

An irregular sampling theorem should satisfy the following requirements:

1. The recovery of a band-limited function from its sampled values is constructive,
2. stable under errors (round-off, truncation, jitter, and aliasing errors),
3. and possible in arbitrary dimensions.
4. The reconstruction method is local, and
5. applies to a general class of norms.

These items are not independent of each other. For instance, reconstruction in stronger norms such as weigted L^1_α requires automatically better decay properties of the expanding functions. Realistic signals and images are usually both almost band-limited and almost time limited, and are thus contained in much smaller spaces than L^2 or L^∞. The utilization of other norms therefore seems desirable in applications, but has been neglected in the literature.

If one wants to make use of these reconstruction procedures based on [27, 10], certain limitations have to be accepted. Both theorems are based on Hilbert space techniques and consequently lack locality. This implies that the reconstruction of a rapidly decreasing function does not converge better than for an arbitrary band-limited function of finite energy. A reconstruction from a finite number of samples will introduce artificial, slowly decaying "tails" of the function outside the sampled interval. In this respect the situation is similar to the convergence properties of the cardinal series, see [6] for a discussion of locality in the regular case.

In our previous papers [14, 17] it is shown how to obtain a reconstruction method that incorporates the additional requirements 1–5. The basic idea for the reconstruction is the approximation of a convolution operator and its subsequent iteration.

In this paper we present a new approach to irregular sampling and a thorough theoretical analysis of the resulting reconstruction algorithms. Similar to [10] they are also based on a perturbation argument, but the starting point is the observation that band-limited functions satisfy a reproducing formula, rather than the classical Shannon–Whittacker sampling theorem.

Progress in this paper is achieved by means of operator theory, avoiding the direct arguments of [17], and the introduction of the local oscillation of a function. The irregular sampling theorem turns out to be equivalent to the factorization of a convolution operator, and the reconstruction thus becomes much more transparent than before. Since no derivatives occur, our new approach allows for an irregular sampling theorem in general locally compact abelian groups (Theorem 4). The main advantage of the new techniques, however, is that stability and error analysis—to be discussed in a subsequent paper—become feasible, quite in contrast to [17], where error analysis seemed to be an awkward task.

Similar ideas have been used for some time in other areas of harmonic analysis to obtain simple decompositions of functions, so-called wavelet and Gabor type expansions, see [8, 19, 15, 16, 20]. The application of these ideas to irregular sampling seems to be new.

A somewhat surprising product of our investigations is the following theorem which comes closest to the spirit of the classical sampling theorem without having its inconveniences:

THEOREM. *Let $\Omega \subseteq R^n$ be compact and $g \in L^1_\alpha$ band-limited with $\hat{g} \neq 0$ on Ω. Then there exist $\delta_1(g)$, $\delta_2(g) > 0$ (depending only on g) such that the following is true: If $X = (x_i)_{i \in I}$ is δ_1-dense and $Y = (y_j)_{j \in J}$ is δ_2-dense, then every $f \in L^p_w$, $1 \leqslant p < \infty$, with* spec $f \subseteq \Omega$ *has the representation*

$$f = \sum_{j \in J} c_j(f(x_i)_{i \in I}) L_{y_j} g \tag{2}$$

with convergence in L^p_w and uniformly on compact sets. The coefficients

c_j depend only on the sampled values $f(x_i)$ and the linear mapping $c: (f(x_i))_{i \in I} \rightarrow (c_j)_{j \in J}$ is continuous from l^p_w into l^p_w.

This theorem combines both aspects of the classical sampling theorem and the cardinal series: it provides a complete reconstruction of f starting from its sampled values and simultaneously a series expansion with respect to translates of an almost arbitrary function g. Moreover, the requirements 2–5 are also satisfied: the series converges in the right norm, and thus all problems that previously occurred with the cardinal series are avoided. Since the calculation of the coefficients from the sampled values is very complicated, this theorem might be only of theoretical interest. But if sampling and series expansions are considered separately, one obtains simpler and more practical versions of the above theorem (see Theorems 1 and 2 in Section 5). These are easily implemented and may be quite useful in future applications.

It might be argued that in Theorems 1 and 2 neither the sampling densities nor the expanding functions are explicitly given and that this missing information could be a serious obstacle for relevant applications. However, one of us has recently shown that for a simplified version of the algorithm any sampling rate higher than the Nyquist rate is sufficient. Sharp estimates for the speed of convergence of that algorithm allow us to find the number of iterations required to obtain a certain accuracy a priori [21].

Since the first draft of this paper, several versions of our algorithms have been implemented and tested. The results have been very convincing: compared to the other iterative reconstruction algorithms mentioned above, the numerical algorithms based on our theoretical results required considerably less iterations for a good reconstruction of a signal. Especially in the case of strong irregularities of the sampling set our procedures seem superior to other methods. See [18, 7] for preliminary numerical results.

The paper is organized as follows. To make it self-contained, Section 2 recalls some notation and contains a few simple lemmata that are frequently used in the sequel. Sections 3 and 4 treat the fundamental steps towards the sampling theorems: the approximation of convolutions (Section 3) and the factorization of convolution operators (Section 4). In Section 5 these results are combined and interpreted, and thus several theorems on irregular sampling and series expansions of band-limited functions are obtained. We shall conclude with remarks on the relation of the sampling density and the size of the spectrum and show some converse results about the necessity of our assumptions. In Section 6 irregular sampling in general locally compact abelian groups is discussed.

The error analysis can be carried out conveniently based on Sections 3–5. It will be the subject of a separate work.

2. Notation and Basic Concepts

In this section the notation is explained and some basic inequalities are stated. Some of them hold true on general locally compact groups [16, 20]. For the sake of clarity we give explicit versions on R^n, where the inequalities are easy consequences of the appropriate definitions, see also [17].

2.1. $(B, \|\cdot\|_B)$ denotes a solid Banach space of functions which is an L^1_α-convolution module, i.e.,

(1) $(B, \|\cdot\|_B)$ is continuously embedded into L^1_{loc},

(2) If $|f(x)| \leqslant |g(x)|$ a.e. and $g \in B$, then $f \in B$ and $\|f\|_B \leqslant \|g\|_B$,

(3) B is invariant under translations $L_x f(y) = f(y-x)$, $x, y \in R^n$, and satisfies $\|L_x f\|_B \leqslant C(1+|x|)^\alpha \|f\|_B$ for some $\alpha \geqslant 0$ and some constant $C > 0$. Writing $L^1_\alpha = \{f \in L^1, f \cdot (1+|x|)^\alpha \in L^1\}$, we require that

$$\begin{gathered} L^1_\alpha * B \subseteq B \\ \|f * g\|_B \leqslant \|f\|_{1,\alpha} \|g\|_B. \end{gathered} \tag{3}$$

It follows from these assumptions that B is embedded into the tempered distributions, thus the Fourier transform is defined at least in the distributional sense.

The reader who is not interested in generality may think of B as a weighted L^p_w-space $L^p_w = \{f \text{ measurable}, \int |f(x)|^p w(x)^p \, dx < \infty\}$, where $w(x) \geqslant 0$ and $w(x+y) \leqslant (1+|x|)^\alpha w(y)$ for all $x, y \in R^n$. Since the arguments do not simplify in these special cases, we keep the notation and treat the general problem.

C_h denotes the convolution operator $f \mapsto C_h f = f * h$. For any compact set $\Omega \subseteq R^n$ we write $B^\Omega = \{f \in B, \operatorname{spec} f = \operatorname{supp} \hat{f} \subseteq \Omega\}$ for the space of band-limited functions in B with spectrum in Ω.

2.2 The *density* of a set $X = (x_i)_{i \in I}$ of sampling points is defined as the infimum of all $\delta > 0$, such that $R^n = \bigcup_{i \in I} B_\delta(x_i)$ (where $B_\delta(x_i)$ is the open ball of radius δ centered at x_i). We shall assume that X is a finite union of discrete sets $X_r = (x_i)_{i \in I_r}$, $r = 1, ..., n$, i.e., $|x_i - x_j| \geqslant \delta_0 > 0$ for all $i, j \in I_r$, $i \neq j$ (cf. [28] for another definition of density).

To a sampling set X we associate a partition of unity $\Psi = (\psi_i)_{i \in I}$ of size δ with the properties

(1) ψ_i measurable and $0 \leqslant \psi_i \leqslant 1$ for all i

(2) $\operatorname{supp} \psi_i \subseteq B_\delta(x_i)$

(3) $\sum_{i \in I} \psi_i \equiv 1$.

For example, in R^n one could take the characteristic function of the so-called Voronoi region around x_i, $V_i = \{x \in R^n : |x - x_i| \leqslant |x - x_j|, \forall j \neq i\}$. In R^2 triangulation and piecewise linear interpolation, as it is used in [38], are simple and quite efficient in numerical computations.

2.3. To describe the local behavior of a function, we shall frequently use the *local maximal function*

$$f^{\#}(x) = \sup_{z \in B_\delta(x)} |f(x)| \tag{4}$$

and the *local oscillation*

$$\operatorname{osc}_\delta f(x) = \sup_{z \in B_\delta(x)} |f(z) - f(x)|. \tag{5}$$

It is clear that $f^{\#} \in B$ if and only if $f \in B$ and $\operatorname{osc}_\delta f \in B$ for one and therefore all $\delta > 0$. The behavior of $^{\#}$ and osc under convolution is given by the inequalities

$$(f * h)^{\#}(x) \leqslant |f| * h^{\#}(x) \tag{6}$$

and

$$\operatorname{osc}_\delta(f * h)(x) \leqslant |f| * \operatorname{osc}_\delta h(x). \tag{7}$$

These inequalities follow immediately by pulling the sup through the integral (compare also [20]).

If $f \in B$ is continuous, $f^{\#} \in B$, and the bounded functions with compact support are dense in B, then $\lim_{\delta \to 0} \operatorname{osc}_\delta f = 0$ holds true pointwise and in B.

2.5. Given a sampling set $X = (x_i)_{i \in I}$ and any associated partition of unity $\Psi = (\psi_i)_{i \in I}$, we consider the following operators:

(a) For continuous f

$$\mathrm{Sp}_\Psi f = \sum_{i \in I} f(x_i) \psi_i \tag{8}$$

may be thought of as an irregular spline approximation of f. From the pointwise estimate $|\mathrm{Sp}_\Psi f| \leqslant f^{\#}$ it is obvious that

$$\|\mathrm{Sp}_\Psi f\|_B \leqslant \|f^{\#}\|_B. \tag{9}$$

(b) The approximation by discrete measures

$$D_\Psi f = \sum_{i \in I} \langle \psi_i, f \rangle \delta_{x_i} \tag{10}$$

makes sense for any locally integrable function $f \in L^1_{\mathrm{loc}}$.

It was shown in [17] that D_Ψ is continuous in the sense that

$$\|D_\Psi f * k\|_B \leqslant C_k \|f\|_B \tag{11}$$

for all bounded functions k of compact support with a constant depending only on k.

(c) Similarly the measure

$$D_\Psi^+ f = \sum_{i \in I} f(x_i) \left(\int \psi_i \right) \delta_{x_i} \tag{12}$$

could be considered for continuous f. Then the following equation is valid:

$$\|D_\Psi^+ f * k\|_B \leqslant C_k \|f^\#\|_B. \tag{13}$$

2.6. The convergence of a sum $\sum_{i \in I} f_i$ in B is unconditional, i.e., for any exhausting sequence $\ldots F_n \subset F_{n+1} \subset \ldots \nearrow I$ of finite subsets of the (countable) index set I, the sequence of partial sums $\sum_{i \in F_n} f_i$ converges to a unique element $f = \sum_{i \in I} f_i$ in the norm of B. Therefore any rearrangement of the sum will converge to f, and this allows us to choose the most convenient summation method for the numerical implementation of the algorithms.

2.7. If $f \in B$ has compact spectrum, then its local maximal function and its oscillation can be easily estimated. The following lemma will occur frequently in our further arguments.

LEMMA 1. *Let $\Omega_0 \subseteq R^n$ be compact. Then for $h \in B^{\Omega_0}$*

$$\|h^\#\|_B \leqslant d(\Omega_0) \|h\|_B \tag{14}$$

and

$$\|\mathrm{osc}_\delta h\|_B \leqslant c(\delta, \Omega_0) \|h\|_B \tag{15}$$

hold true. Here $c(\delta, \Omega_0) = \inf \|\mathrm{osc}_\delta p\|_{1,\alpha} = O(\delta)$, where the infimum ranges over all functions $p \in L^1_\alpha$ with $\hat{p}(t) = 1$ on Ω_0.

Proof. We write $h = h * p$ with some band-limited function $p \in L^1_\alpha$ with $\hat{p}(t) = 1$ on Ω_0. Then (14) is a consequence of (6) and $d(\Omega_0)$ may be taken to be $d(\Omega_0) = \inf_p \|p^\#\|_{1,\alpha}$. Similarly, (15) follows from (7). The constant $c(\delta, \Omega_0)$ is of order δ, because

$$\mathrm{osc}_\delta p(x) \leqslant \sup_{z \in B_\delta(x)} |z - x| \, |\nabla p(z')| \leqslant \delta \sup_{z \in B_\delta(x)} |\nabla p(z)|$$

and $\|\mathrm{osc}_\delta p\|_{1,\alpha} \leqslant \delta \, \| \, |\nabla p|^\# \|_{1,\alpha}$. ∎

Finally we observe once and for all that convergence of a sequence of band-limited functions f_n in B implies convergence in a finer norm.

LEMMA 2. *Set $CB = \{f \in B : f$ continuous and $f^\# \in B\}$ with norm $\|f\|_{CB} = \|f^\#\|_B$. If $f_n \in B^\Omega$ converges to $f \in B^\Omega$ in the B-norm, then it converges in CB. In particular, f_n converges to f uniformly on compact sets.*

Proof. As before we choose $p \in L^1_\alpha$ with $\hat{p}(t) = 1$ on Ω. Then $f_n * p = f_n$, $f * p = f$ and

$$\|f - f_n\|_{CB} = \|((f - f_n) * p)^\#\|_B \leqslant \|f - f_n\|_B \, \|p^\#\|_{1,\alpha}$$

which proves the lemma.

Remark. Lemmas 1 and 2 assert that $B^{\Omega_0} \subseteq CB$ and that on B^{Ω_0} the norms of B and CB are equivalent, compare also [17].

3. APPROXIMATION OF CONVOLUTIONS

In this section we analyze the first step towards sampling and expansion theorems for band-limited functions, namely the approximation of certain convolution operators. Similar results hold true in non-abelian groups [20]. In the sequel we shall try to approximate the convolution operator $C_h f$ on the subspaces B^Ω of B by operators which need only the sampled values of f as input or by operators using only translates of h. The use of the oscillation makes most estimates look very simple, but the reader should be aware that it is essentially a substitute for Bernstein's inequality, which is not even available in the literature in the generality required. For a different approach see [17]

Let $X = (x_i)_{i \in I}$ and $Y = (y_j)_{j \in J}$ be two sets of sampling points and $\Psi = (\psi_i)_{i \in I}$ and $\Phi = (\phi_j)_{j \in J}$ be two partitions of unity of size δ_1 and $\delta_2 > 0$ associated to X and Y and assume that $h \in CL^1_\alpha$, i.e., both h, $h^\# \in L^1_\alpha$. It is clear that any Schwartz function and any band-limited function in L^1_α has this property.

With these assumptions we consider the following approximations of C_h:

$$A_1 f = (\mathrm{Sp}_\Psi f) * h = \sum_{i \in I} f(x_i) \psi_i * h \tag{16}$$

$$A_2 f = (D_\Psi f) * h = \sum_{i \in I} \langle \psi_i, f \rangle \, L_{x_i} h \tag{17}$$

$$A_3 f = (D^+_\Psi f) * h = \sum_{i \in I} f(x_i) \left(\int \psi_i \right) L_{x_i} h \tag{18}$$

$$A_4 f = [D_\Phi(\mathrm{Sp}_\Psi f)] * h = \sum_{i \in I, j \in J} f(x_i) \left(\int \psi_i \phi_j \right) L_{y_j} h. \tag{19}$$

If both $h_1, h_2 \in CL^1_\alpha$, then we can consider the following approximation of the product $C_{h_1}C_{h_2} = C_{h_1 * h_2}$:

$$A_5 f = [D_\Phi(\mathrm{Sp}_\Psi f * h_1)] * h_2$$
$$= \sum_{i \in I, j \in J} f(x_i)\left(\int \psi_i * h_1(z)\phi_j(z)\, dz\right) L_{y_j} h_2. \tag{20}$$

To indicate the dependence of these operators on h, X, Ψ, Φ, we shall sometimes use the more precise notation $A_i(h, X, \Psi)$. As a consequence of the basic estimates in Section 2 all these operators are well defined on either B or the subspaces B^Ω. Of course, various other approximations can be taken into consideration, but the operators A_i are all we need. The following two lemmata show in which sense these operators approximate C_h on B or on B^Ω. If the sampling values of f are involved, the estimates are only valid on the subspace $CB = \{f \in B, f^\# \in B\}$ of B.

To assure norm convergence of all appearing sums $\sum_{i \in I}$, we shall assume from now on that the bounded functions with compact support are dense in B. Then the partial sums of $\sum_{i \in I} f(x_i)\psi_i$, say, are norm convergent. Thus, if we apply a continuous operation to such a sum, C_h, for instance, we can always interchange the sum $\sum_{i \in I}$ and the operation, e.g.,

$$(\mathrm{Sp}_\Psi f) * h = \sum_{i \in I} f(x_i)(\psi_i * h).$$

Hence, all formal interchanges are well justified.

LEMMA 3. *Assume that $h \in L^1_\alpha$ has compact spectrum* $\mathrm{spec}\, h \subseteq \Omega_0$. *Then*

$$\|C_h f - A_i f\|_B \leq c(\delta, \Omega_0)\, \|h\|_{1,\alpha}\, \|f\|_B, \qquad i = 1, 2 \tag{21}$$

and

$$\|C_h f - A_3 f\|_B \leq c(\delta, \Omega_0)(1 + d(\Omega_0))\, \|h\|_{1,\alpha}\, \|f\|_B \tag{22}$$

hold true for all $f \in B^{\Omega_0}$.

Proof. *Case* A_1. From

$$|f - \mathrm{Sp}_\Psi f| \leq \mathrm{osc}_\delta f \quad \text{a.e.} \tag{23}$$

we obtain

$$\|C_h f - A_1 f\|_B = \|(f - \mathrm{Sp}_\Psi f) * h\|_B \leq \|\mathrm{osc}_\delta f * |h|\,\|_B$$
$$\leq c(\delta, \Omega_0)\, \|f\|_B\, \|h\|_{1,\alpha} \tag{24}$$

as a consequence of (3) and Lemma 1.

Case A_2. Consider one summand of $(\sum \psi_i f) * h - \sum \langle \psi_i, f \rangle L_{x_i} h$ and estimate

$$\left| \int \psi_i(y) f(y)(h(x-y) - h(x-x_i))\, dy \right|$$
$$\leqslant \int \psi_i(y)\, |f(y)|\, \operatorname{osc}_\delta h(x-y)\, dy. \tag{25}$$

Summing over I we find

$$|f * h - D_\Psi * h| \leqslant |f| * \operatorname{osc}_\delta h \tag{26}$$

pointwise and thus by Lemma 1

$$\|C_h f - A_2 f\|_B \leqslant \|f\|_B \, \|\operatorname{osc}_\delta h\|_{1,\alpha} \leqslant c(\delta, \Omega_0)\, \|h\|_{1,\alpha} \, \|f\|_B. \tag{27}$$

Case A_3. We observe that $C_h - A_3 = C_h - A_1 + A_1 - A_3$ and

$$|A_1 f(x) - A_3 f(x)| = \left| \sum_i f(x_i) \int \psi_i(y)[h(x-y) - h(x-x_i)]\, dy \right|$$
$$\leqslant \sum_{i \in I} |f(x_i)| \int \psi_i(y) \operatorname{osc}_\delta h(x-y)\, dy$$
$$\leqslant (\mathrm{Sp}_\Psi |f|) * \operatorname{osc}_\delta h(x). \tag{28}$$

Taking norms and using Lemma 1, (3) and (9) yield

$$\|A_1 f - A_3 f\|_B \leqslant \|\mathrm{Sp}_\Psi |f|\,\|_B \, \|\operatorname{osc}_\delta h\|_{1,\alpha}$$
$$\leqslant \|f^\#\|_B \, c(\delta, \Omega_0)\, \|h\|_{1,\alpha} \leqslant c(\delta, \Omega_0) d(\Omega_0)\, \|h\|_{1,\alpha}\, \|f\|_B. \tag{29}$$

Consequently we obtain

$$\|C_h f - A_3 f\|_B \leqslant \|C_h f - A_1 f\|_B + \|A_1 f - A_3 f\|_B$$
$$\leqslant c(\delta, \Omega_0)(1 + d(\Omega_0))\, \|h\|_{1,\alpha}\, \|f\|_B. \quad \blacksquare \tag{30}$$

LEMMA 4. *Suppose that h, h_1, $h_2 \in L^1_\alpha$ are band-limited with common spectrum Ω_0. Then for all $f \in B^{\Omega_0}$*

$$\|C_h f - A_4 f\|_B \leqslant (c(\delta_1, \Omega_0) + c(\delta_2, \Omega_0) d(\Omega_0))\, \|h\|_{1,\alpha}\, \|f\|_B \tag{31}$$

and

$$\|C_{h_1 * h_2} f - A_5 f\|_B \leqslant (c(\delta_1, \Omega_0) + c(\delta_2, \Omega_0) d(\Omega_0))\, \|h_1\|_{1,\alpha}\, \|h_2\|_{1,\alpha}\, \|f\|_B. \tag{32}$$

Proof. *Case* A_4. We observe that $C_h - A_4 = C_h - A_1 + A_1 - A_4$ and that $A_1 f - A_4 f = C_h \operatorname{Sp}_\Psi f - A_2(h, Y, \Phi) \operatorname{Sp}_\Psi f$. Therefore (27) with f replaced by $\operatorname{Sp}_\Psi f$ implies

$$\begin{aligned}\|A_1 f - A_4 f\|_B &\leqslant c(\delta_2, \Omega_0)\, \|h\|_{1,\alpha}\, \|\operatorname{Sp}_\Psi f\|_B \\ &\leqslant c(\delta_2, \Omega_0)\, d(\Omega_0)\, \|h\|_{1,\alpha}\, \|f\|_B. \end{aligned} \tag{33}$$

Combining (24) and (33) yields finally

$$\|C_h f - A_4 f\|_B \leqslant (c(\delta_1, \Omega_0) + c(\delta_2, \Omega_0) d(\Omega_0))\, \|h\|_{1,\alpha}\, \|f\|_B. \tag{34}$$

Case A_5. Note that in the full notation $A_5 = A_2(h_2, Y, \Phi) A_1(h_1, X, \Psi)$. Therefore

$$\begin{aligned}&\|C_{h_1} C_{h_2} f - A_2(\ldots) A_1(\ldots) f\|_B \\ &\quad\leqslant \|C_{h_2}(C_{h_1} f - A_1 f)\|_B + \|(C_{h_2} - A_2) A_1 f\|_B \\ &\quad\leqslant \|C_{h_1} f - A_1 f\|_B\, \|h_2\|_{1,\alpha} + c(\delta_2, \Omega_0)\, \|h_2\|_{1,\alpha}\, \|A_1 f\|_B \\ &\quad\leqslant c(\delta_1, \Omega_0)\, \|h_1\|_{1,\alpha}\, \|h_2\|_{1,\alpha}\, \|f\|_B \\ &\quad\quad + c(\delta_2, \Omega_0)\, \|h_2\|_{1,\alpha} d(\Omega_0)\, \|f\|_B\, \|h_1\|_{1,\alpha}. \end{aligned}$$

Here we have used Lemma 1 several times, Cases A_1, A_2 of Lemma 3 and the estimate

$$\|\operatorname{Sp}_\Psi f * h\|_B \leqslant \|f^\#\|_B\, \|h\|_{1,\alpha}. \quad \blacksquare \tag{35}$$

We may now summarize the results on the approximation of convolution operators:

Proposition 5. *Given h, h_1, $h_2 \in L^1_\alpha$ with common spectrum in Ω_0. Then the A_i are bounded operators from B^{Ω_0} into B^{Ω_0}. Moreover, A_2 is a continuous operator from B into B^{Ω_0} and A_1, A_3, A_4, A_5 from $CB = \{f \in B, f^\# \in B\}$ into B^{Ω_0}. If δ, δ_1, δ_2 are chosen such that*

$$\begin{aligned} c(\delta, \Omega_0) &< \|h\|_{1,\alpha}^{-1} \qquad \textit{for} \quad i = 1, 2, \\ c(\delta, \Omega_0)(1 + d(\Omega_0)) &< \|h\|_{1,\alpha}^{-1} \qquad \textit{for} \quad i = 3, \end{aligned} \tag{36}$$

and

$$(c(\delta_1, \Omega_0) + c(\delta_2, \Omega_0) d(\Omega_0))\, \|h_1\|_{1,\alpha}\, \|h_2\|_{1,\alpha} < 1, \qquad i = 4, 5, \tag{37}$$

respectively, then the operator norms satisfy

$$|||C_h - A_i|||_{B^{\Omega_0} \to B^{\Omega_0}} < 1$$

for $i = 1, 2, 3, 4$ *and*

$$|||C_{h_1 * h_2} - A_5||| < 1.$$

Consequently for δ small enough the geometric series

$$D_i = \sum_{n=0}^{\infty} (C_h - A_i)^n, \qquad i = 1, 2, 3, 4 \tag{38}$$

and

$$D_5 = \sum_{n=0}^{\infty} (C_{h_1 * h_2} - A_5)^n \tag{39}$$

are well defined bounded operators from B^{Ω_0} into B^{Ω_0}.

Remark. The conditions on δ look rather unpractical and implicit, but according to Lemma 1 they may be replaced by the weaker estimate

$$\delta < (\|\mathrm{osc}_\delta\, p\|_{1,\alpha}\, \|h\|_{1,\alpha})^{-1} \tag{40}$$

for any function $p \in L^1_\alpha$ with $\hat{p}(t) = 1$ on Ω_0 (in the case of A_1 and A_2). Analogous substitutions can be made in the case of A_3, A_4, A_5.

4. Factorization of Convolutions

Once we have found good approximations to the convolution operator C_h, we are able to start an iteration, i.e., build a first error term $f - Af$, apply A again to obtain the first correction and then continue. It is not at all clear, whether such an iteration produces any resonable result, because starting with $f \in B^\Omega$ already in the first step we leave this space and end up in the larger space B^{Ω_0}. This can be corrected by using another auxiliary function, see [14]. Instead of the rather tricky argument of [14] we will present an abstract argument which makes the iteration entirely transparent. It leads to the factorization of a convolution operator which might be of independent interest.

Given $g \in L^1_\alpha$ band-limited we choose a band-limited function $h \in L^1_\alpha$ with $\Omega_0 := \mathrm{spec}\, h \supseteq \mathrm{spec}\, g$ and $\hat{h}(x) = 1$ on $\mathrm{spec}\, g$. This entails the relations

$$g * h = g \qquad \text{and} \qquad C_g C_h = C_g. \tag{41}$$

Let the bounded linear operator A be any good approximation of C_h on B^{Ω_0}, e.g., one of the operators $A_i(h, X, \Psi)$, and assume that the remainder $R := C_h - A$ is a contraction on B^{Ω_0}, i.e.,

$$|||R|||_{B^{\Omega_0} \to B^{\Omega_0}} = |||C_h - A|||_{B^{\Omega_0}} < 1. \tag{42}$$

Then $D := \sum_{k=0}^{\infty} R^k$ is well defined on B^{Ω_0}.

PROPOSITION 6. *Under the assumptions stated, C_g factorizes as follows*:

1. *On any subspace of B which is mapped into B^{Ω_0} by A*

$$C_g = C_g AD. \tag{43}$$

2. *On all of B*

$$C_g = DAC_g. \tag{44}$$

3. *If $f \in B$ and* $\operatorname{spec} f \subseteq \operatorname{spec} g$, *then also*

$$C_g = C_g DA \tag{45}$$

holds true.

Proof. We use (41) and the trivial formula

$$C_g C_h = C_g A + C_g R. \tag{46}$$

In (46), $C_g A$ is a "good" approximation of C_g and $C_g R$ is the remainder term. Now (46) allows us to iterate this argument.

$$C_g = C_g C_h = C_g A + C_g R = C_g A + C_g AR + C_g R^2$$

and by induction we obtain

$$C_g = C_g A \sum_{k=0}^{n} R^k + C_g R^{n+1}. \tag{47}$$

Since R is a contraction on B^{Ω_0}, we obtain the factorization of C_g

$$C_g = C_g A \sum_{k=0}^{\infty} R^k = C_g AD \tag{48}$$

by letting $n \to \infty$. If A maps a subspace of B into B^{Ω_0}—possibly all of B as in the case of the approximation operator A_2—then so does R and there is no problem to define D on this subspace. Thus (43) holds also on this subspace.

For (2) we use the fact that $C_g = C_h C_g$ and $C_h C_g = AC_g + RC_g$. Repeating the same argument as above we obtain

$$C_g = \left(\sum_{k=0}^{\infty} R^k \right) AC_g = DAC_g. \tag{49}$$

Since C_g maps B into B^{Ω_0} and both A and D are defined there, (44) is valid on the entire space B.

Proof of (3). Set $\Omega_1 := \operatorname{spec} g$. Then we show that on B^{Ω_1} the equality

$$C_g AD = C_g DA \tag{50}$$

holds. For this we show by induction that

$$C_g A(C_h - A)^k = C_g(C_h - A)^k A \tag{51}$$

is true for all $k \geqslant 0$ on B^{Ω_1}. This is obvious for $k = 0$, therefore we may assume that (51) is true for $k \leqslant n$. Then for $f \in B^{\Omega_1}$

$$\begin{aligned} C_g(C_h - A)^{n+1} Af &= C_g C_h(C_h - A)^n Af - C_g A(C_h - A)^n Af \\ &= C_g A(C_h - A)^n C_h f - C_g A(C_h - A)^n Af \\ &= C_g A(C_h - A)^{n+1} f. \end{aligned}$$

In the middle equality we have used the induction hypothesis and the fact that $C_h f = f$ for $f \in B^{\Omega_1}$. ∎

5. Irregular Sampling Theorems and Series Expansions for Band-Limited Functions

At this point it takes only a little effort to deduce very general theorems on sampling and on series expansions of band-limited functions. All we have to do is to choose an appropriate approximation of the convolutions involved and interprete its factorization. Weaker versions of Theorems 1 and 2 have already been obtained in [17] in a different way. Our new approach seems to be more satisfactory for several reasons: (a) The nature of the reconstruction is now more transparent. (b) With only minor modifications irregular sampling theorems for locally compact abelian groups can be proven, see Theorem 4. (c) The factorization of the convolution facilitates the stability and error analysis of the reconstruction which seemed to be intractable with the methods of [17].

Theorem 3 combines both the sampling and the series expansion aspect. Whereas in its general form it may not be useful in applications—in contrast to Theorems 1 and 2— it shows how far one can go in a theory of irregular sampling that still satisfies all required properties of stability, locality, etc.

In the following theorems $(B, \|\cdot\|_B)$ is always a Banach space of functions that satisfies the general assumptions in 2.1 and which contains the bounded functions of compact support as a dense subspace. The spectrum $\Omega \subseteq R^n$ will always be a compact set.

THEOREM 1. *Choose a band-limited function* $h \in L^1_\alpha$ *with* spec $h = \Omega_0 \supseteq \Omega$ *such that* $\hat{h}(t) = 1$ *on an open neighborhood of* Ω. *Choose* $\delta > 0$ *such that* $c(\delta, \Omega_0) < \|h\|^{-1}_{1,\alpha}$. *Then any* $f \in B^\Omega$ *can be completely reconstructed from its sampled values on any* δ*-dense subset* $X = (x_i)_{i \in I}$ *and there are functions* $e_i \in L^1_\alpha$ *with* spec $e_i \subseteq \Omega_0$ *such that*

$$f = \sum_{i \in I} f(x_i) e_i. \tag{52}$$

Here the sum converges in B and uniformly on compact sets.

Remarks. 5.1. From a functional analytic point of view Theorem 1 may also be formulated in the following way: There exist "correction" operators D_i, bounded on B^{Ω_0}, which recover the band-limited functions $f \in B^\Omega$ from the approximations A_i, $i = 1, 3, 4, 5$ of f by $f = D_i A_i f$, provided that the sampling rate δ is small enough.

5.2. The proof establishes the following algorithm for the reconstruction of f. Starting with $f(x_i)$, set

$$\phi_0 = \sum_{i \in I} f(x_i) \psi_i * h \tag{53}$$

as the first approximation of f. The nth correction term is defined inductively by

$$\phi_n = \phi_{n-1} * h - \sum_{i \in I} \phi_{n-1}(x_i) \psi_i * h. \tag{54}$$

Then finally

$$f = \sum_{n=0}^{\infty} \phi_n. \tag{55}$$

For numerical work the approximation operator A_3 of (18) is preferable, because only the $f(x_i)$ and the $h * h * \cdots * h(x_i)$ are needed, but no other convolutions.

5.3. The choice of $h \in L^1_\alpha$ and $\Omega_0 = \text{spec}\, h$ is only necessary to specify the density of the sampling in terms of the ingredients of the reconstruction. If Ω is an interval of length $2\pi W$, then the sampling rate δ is necessarily smaller than the Nyquist rate $1/W$ [29]. For a simplified version of the algorithm (53)–(55) with $h(x) = \sin Wx/Wx$ and $B = L^2(R)$ it is shown in [21] that any sampling rate $\delta < 1/W$ is sufficient for the convergence of the algorithm. In view of this result it is reasonable to expect that for a suitable choice of h the required sampling rate is close to the Nyquist rate also in the general case.

5.4. Since the functions $e_i \in L^1_\alpha$ are band-limited and hence in CL^1_α, it is easily seen that they decay at infinity like $o((1+|x|)^{-\alpha})$. It is now clear that the localization of the e_i's is much better than in previous reconstructions [10, 27, 45], where only $e_i \in L^2$ is known, or even in the cardinal series.

Proof of Theorem 1. Since $\hat{h}=1$ on an open neighborhood of Ω, Ω' say, we can find a function $g \in L^1_\alpha$ such that $\hat{g}(x)=1$ on Ω and spec $g \subseteq \Omega'$. This guarantees

$$g * h = g \qquad \text{or} \qquad C_g C_h = C_g. \tag{56}$$

For the purpose of sampling, any approximation of C_h that contains the sampled values $f(x_i)$ only, e.g., A_1, A_3, A_4, will do the job. If $X=(x_i)_{i \in I}$ is δ-dense, then $|||C_h - A_1(h)||| < 1$ on B^{Ω_0} by Proposition 5. Now Proposition 6 applies and provides the factorization of C_g,

$$C_g = D_1 A_1 C_g \quad \text{on } B \qquad \text{or} \qquad C_g = C_g D_1 A_1 \quad \text{on } B^{\Omega}. \tag{57}$$

But written out in detail, this is nothing but the desired sampling theorem. For $f \in B^{\Omega}$, $f * g = f$ still holds and

$$f = C_g f = D_1 A_1 C_g f = D_1 \left(\sum_{i \in I} f(x_i) \psi_i * h \right) = \sum_{i \in I} f(x_i) D_1(\psi_i * h). \tag{58}$$

Since D_1 is continuous on B^{Ω_0}, the latter sum indeed converges in B. The argument depends only on the translation norm $(1+|x|)^\alpha$ of B and thus it is also valid in L^1_α. Consequently the $e_i = D_1(\psi_i * h)$ are in L^1_α and spec $e_i \subseteq$ spec $h = \Omega_0$. The same argument applies to the operators A_3 and A_4. ∎

Remark 5.5. Note that from (58) we obtain the estimate

$$\|f\|_B \leqslant |||D_1||| \; \|A_1 f\|_B \leqslant |||D_1||| \; \|h\|_{1,\alpha} \left\| \sum_{i \in I} f(x_i) \psi_i \right\|_B \tag{59}$$

which shows that the reconstruction depends continuously on the sampling values. If $B = L^p_w$ then it is easy to see, cf. [17] that $\|\mathrm{Sp}_\Psi f\|_{p,w}$ is equivalent to the discrete l^p_w-norm $(\sum_{i \in I} |f(x_i)|^p \, w(x_i)^p)^{1/p}$. Thus (59) is in fact the natural estimate for the continuous dependence of f on its sampled values and expresses that the sampling is *stable*, cf. [23, p. 83] or [29].

Remark 5.6 (Bunched Sampling or Periodic Sampling [44, 34]). Assume that (in one dimension) the sampling X is periodic, i.e., $x_{i+nr} = n\alpha + \delta_i$, $i = 0, 1, ..., r-1$, $n \in N$. In other words, X has the symmetries $L_{n\alpha}$.

Then the e_i can be constructed to have the same symmetries: $e_{i+nr} = L_{n\alpha} e_i$ and only r basis functions need to be calculated.

Proof. If we choose $\psi_i = c_{[x_{i-1}/2 + x_i/2,\, x_i/2 + x_{i+1}/2]}$ as the partition of unity associated to X, then $\psi_{i+nr} = L_{n\alpha}\psi_i$ and $L_{k\alpha} A_1 f = \sum_{i,n} f(x_{i+nr}) L_{(k+n)\alpha}\psi_i * h = A_1 L_{k\alpha} f$ for all $k \in N$. Since A_1 commutes with $L_{k\alpha}$, both $(C_h - A_1)^n$, $n \geqslant 0$ and D_1 also commute with all $L_{k\alpha}$. Therefore $e_{i+nr} = D_1(\psi_{i+nr} * h) = D_1(L_{n\alpha}\psi_i * h) = L_{n\alpha} D_1(\psi_i * h)$ and it suffices to compute the r functions $e_0, e_1, ..., e_{r-1}$ only. The same argument applies to symmetries in higher dimensions, if we use approximation operators that commute with the symmetry.

Remark 5.7. Given $\Omega = [-\pi, \pi]$ and $h \in L^1(R)$, such that $\hat{h}(t) = 1$ for $|t| \leqslant \pi$ and $\hat{h}(t) = 0$ for $|t| > \alpha > \pi$, choose $0 < \beta \leqslant 2\pi/(\alpha + \pi)$. Then the approximation operator A_3 for the regular sampling set $x_n = n\beta$, $n \in Z$ and the partition of unity $\psi_n = \chi_{[\beta(n-1/2),\, \beta(n+1/2)]}$ becomes

$$A_3 f = \beta \sum_{n=-\infty}^{\infty} f(\beta n) L_{\beta n} h.$$

After taking the Fourier transform on both sides and applying Poisson's formula, one finds $A_3 f = f$. Thus only one iteration of the modified algorithm (53)–(55) is required for the complete reconstruction. The well-known versions of the classical regular sampling theorem with fast decreasing kernels, e.g., [6, Theorem 4.3] are therefore special cases of Theorem 1.

THEOREM 2. *Let $g \in L^1_\alpha$ be an arbitrary band-limited function such that $\hat{g}(x) \neq 0$ on Ω. Choose a band-limited function $h \in L^1_\alpha$ with* spec $h = \Omega_0$ *such that $\hat{h}(x) = 1$ on* spec g. *If δ is so small that $c(\delta, \Omega_0) < \|h\|^{-1}_{1,\alpha}$ and if $Y = (y_j)_{j \in J}$ is an arbitrary δ-dense set in R^n, every $f \in B^\Omega$ has a representation*

$$f = \sum_{j \in J} c_j(f)\, L_{y_j} g. \tag{60}$$

The mapping of f to its coefficients is linear and depends continuously on f in the following sense:

$$\left\| \sum_{j \in J} c_j(f)\, \psi_j \right\|_B \leqslant C\, \|f\|_B. \tag{61}$$

Remark. If $B = L^p_w$, then (60) just expresses that c is a continuous linear mapping from L^p_w into l^p_w (compare [17]).

Proof. We can use any approximation of C_h that uses pure translates $L_{y_j}h$ only, e.g., A_2, A_3, A_4. Under the stated assumption on δ and Y Propositions 5 and 6.1 assert the factorization of C_g as $C_g = C_g A_2 D_2$. Since $\hat{h}(t) = 1$ on spec g, which is an open neighborhood of Ω, we can find a local inverse for g, i.e., a band-limited function $g_1 \in L^1_\alpha$ with spec $g_1 \subseteq$ spec g and

$$\hat{g}_1 \cdot \hat{g} \equiv 1 \qquad \text{on } \Omega \tag{62}$$

(theorem of Wiener–Levy, cf. [37, Chap. I]). This means that $f = f * g_1 * g$ and (62) yields the desired representation

$$\begin{aligned} f = C_g C_{g_1} f = C_g A_2 D_2 C_{g_1} f = C_g \left(\sum_j \langle \psi_j, D_2 C_{g_1} f \rangle L_{y_j} h \right) \\ = \sum_{j \in J} \langle \psi_j, D_2 C_{g_1} f \rangle L_{y_j} g \end{aligned} \tag{63}$$

with coeffcients

$$c_j(f) = \langle \psi_j, D_2(f * g_1) \rangle. \tag{64}$$

The continuity (61) follows with the usual arguments (cf. [17]): Set $F = D_2 C_{g_1} f$, then $\|F\|_B \leqslant |||D_2||| \; \|g_1\|_{1,\alpha} \; \|f\|_B$. For a fixed $x \in R^n$ the sum $\sum_j \langle \psi_j, F \rangle \psi_j$ in (61) is finite and contains only those summands where $x \in B_\delta(y_j)$ ($\Leftrightarrow y_j \in B_\delta(x)$). If k is the characteristic function of $B_{2\delta}(0)$, then $|\langle \psi_j, F \rangle \psi_j(x)| \leqslant \langle L_x k, |F| \rangle \psi_j(x)$ and therefore

$$\begin{aligned} \sum_{j \in J} \langle \psi_j, F \rangle \psi_j(x) &\leqslant \sum_j \langle L_x k, |F| \rangle \psi_j(x) \\ &= \langle L_x k, |F| \rangle = |F| * k(x). \end{aligned} \tag{65}$$

We take the B-norm and complete the proof with

$$\left\| \sum_j \langle \psi_j, F \rangle \psi_j \right\|_B \leqslant \| \, |F| * k \, \|_B \leqslant \|F\|_B \, \|k\|_{1,\alpha}. \quad \blacksquare \tag{66}$$

The next theorem combines the features of Theorems 1 and 2. It allows us to represent band-limited functions f as series with respect to translates of a single, almost arbitrary function g and with coefficients that are determined by a sampled version of f. In this sense Theorem 3 is a far-reaching generalization of the classical Shannon–Whittacker sampling theorem.

THEOREM 3. *Let $\Omega \subseteq R^n$ be compact, B as in Theorem 1 and $g \in L^1_\alpha$ band-limited with $\hat{g} \neq 0$ on Ω. Then there exist $\delta_1, \delta_2 > 0$ (depending only on the auxiliary parameters of the construction) such that the following is true:*

If $X = (x_i)_{i \in I}$ is δ_1-dense and $Y = (y_j)_{j \in J}$ is δ_2-dense, then every $f \in B^\Omega$ has the representation

$$f = \sum_{j \in J} c_j(f(x_i)_{i \in I}) L_{y_j} g \tag{67}$$

where the series converges in B and uniformly on compact sets. The coefficients c_j depend only on the sampled values $f(x_i)$ and the linear mapping $c\colon (f(x_i))_{i \in I} \to (c_j)_{j \in J}$ is continuous in sense

$$\left\| \sum_j c_j \phi_j \right\|_B \leqslant C' \, \|f\|_B. \tag{68}$$

Remark 5.8. If $B = L^p_w$, (68) is equivalent to the continuity of c from l^p_w into l^p_w (cf. [17]).

In all three theorems the convergence is actually in CB by Lemma 2.

Proof. In order to apply the techniques of Sections 3 and 4, we have to construct several auxiliary functions first:

Since $\hat{g}(t) \neq 0$ on Ω, we may find a band-limited "extension" $g_1 \in L^1_\alpha$ that satisfies

$$\hat{g}_1(t) = \hat{g}(t) \qquad \text{on } \Omega \qquad \text{and} \qquad \hat{g}_1(t) \neq 0 \qquad \text{on spec } g.$$

If p_Ω denotes a plateau function $p_\Omega \equiv 1$ on Ω and $0 \leqslant p_\Omega \leqslant 1$, then $\hat{g}_1 = p_\Omega \cdot \hat{g} + (p_{\operatorname{spec} g} - p_\Omega) \cdot \xi$ for an appropriate function ξ will work.

By the theorem of Wiener–Levy [37, Chap. I] there is a local inverse $h_1 \in L^1_\alpha$, band-limited such that

$$\hat{h}_1(t)\, \hat{g}_1(t) \equiv 1 \qquad \text{on } \Omega.$$

Now set

$$h_0 = h_1 * g \in L^1_\alpha. \tag{69}$$

Since

$$\hat{h}_0 = \frac{\hat{g}}{\hat{g}_1} = \begin{cases} 1 & \text{on } \Omega \\ 0 & \text{outside spec } g \end{cases} \tag{70}$$

this implies

$$C_{h_0} f = f * h_0 = f \qquad \text{for} \quad f \in B^\Omega \tag{71}$$

and

$$g_1 * h_0 = g. \tag{72}$$

Finally, by Wiener–Levy again, choose a band-limited function $h_2 \in L^1_\alpha$ such that $\hat{g}_1 \cdot \hat{h}_2 \equiv 1$ on spec g. This entails

$$h_0 * g_1 * h_2 = h_0. \tag{73}$$

Now we are in the position to use Lemma 4 and Proposition 6.

Set $\Omega_0 = \operatorname{spec} g_1 \cup \operatorname{spec} h_2$ and choose $\delta_1, \delta_2 > 0$ such that

$$(c(\delta_1, \Omega_0) + c(\delta_2, \Omega_0) d(\Omega_0)) \, \|g_1\|_{1,\alpha} \, \|h_2\|_{1,\alpha} < 1. \tag{74}$$

Then for any δ_1-dense set $X = (x_i)_{i \in I}$ and δ_2-dense set $Y = (y_j)_{j \in J}$ with associated partitions of unity $\Psi = (\psi_i)_{i \in I}$ and $\Phi = (\phi_j)_{j \in J}$ the operator

$$A_5 f = D_\Phi(\mathrm{Sp}_\Psi f * h_2) * g_1 \tag{75}$$

of (20) is a good approximation of $C_{h_2 * g_1}$ on B^{Ω_0} by Propositions 5 and 6.

Consequently Proposition 6 implies the following factorization of C_{h_0}:

$$C_{h_0} = C_{h_0} A_5 D_5. \tag{76}$$

Because of (71) and (72) we obtain

$$\begin{aligned} f = C_{h_0} f = C_{h_0} A_5 D_5 f &= \left(\sum_{i,j} (D_5 f)(x_i) b_{ij} L_{y_j} g_1 \right) * h_0 \\ &= \sum_j \left(\sum_i (D_5 f)(x_i) b_{ij} \right) L_{y_j} g \end{aligned} \tag{77}$$

as desired, where we have abbreviated $\int \psi_i * h_2(y) \phi_j(y) \, dy := b_{ij}$.

Next we show the continuity of the coefficients

$$c_j = \sum_i (D_5 f)(x_i) \, b_{ij}. \tag{78}$$

Because $\sum_j c_j \phi_j = \sum_j \langle \phi_j, \mathrm{Sp}_\Psi(D_5 f) * h_2 \rangle \phi_j$, it can be estimated as in (65) by

$$\begin{aligned} \left\| \sum_j c_j \phi_j \right\|_B &\leqslant C''' \, \|\mathrm{Sp}_\Psi(D_5 f) * h_2\|_B \leqslant C'' \, \|h_2\|_{1,\alpha} \, \|\mathrm{Sp}_\Psi D_5 f\|_B \\ &\leqslant C' \, \|D_5 f\|_B \leqslant C \, \|f\|_B \end{aligned} \tag{79}$$

because $D_5 f \in B^{\Omega_0}$, (9) applies, and D_5 is continuous on B^{Ω_0} by Proposition 5.

It remains to be shown that the coefficients or rather $(D_5 f)(x_i)_{i \in I}$ depend only on the sampling values $f(x_i)$. Since $D_5 = \sum_{n=0}^\infty (C_{h_2 * g_1} - A_5)^n$, it suffices to show this claim for the individual terms $(C_{h_2 * g_1} - A_5)^n f$, which, in turn, is a sum of $C^{n_1}_{h_2*g_1}A^{n_2}_5C^{n_3}_{h_2*g_1}A^{n_4}_5\cdots A^{n_{r-1}}_5C^{n_r}_{h_2*g_1}f$ with $n_1+n_2+\cdots+n_r=n$. Except for the case $n_1=n$, i.e., $C^n_{h_2*g_1}f=f$, all terms contain at least one A_5, i.e., $n_{r-1}\neq 0$ and $\cdots A^{n_{r-1}}_5C^{n_r}_{h_2*g_1}f=\cdots A^{n_{r-1}}_5 f$ contains only information on f derived from the sampling values at x_i. Thus $(D_5 f(x_i))_{i\in I}$ is completely determined by $f(x_i)$ and the proof is complete. ∎

Remark 5.9. The careful reader will observe that the method is not restricted to function spaces B, which are embedded into the tempered distributions. If we accept working with ultradistributions rather than with tempered ones and with general Beurling algebras L^1_w with Beurling–Domar weights w, then all theorems remain true for solid translation-invariant function spaces B, on which such a Beurling algebra L^1_w acts by convolution.

Remark 5.10. The *dependency of the sampling density on the size and geometric shape of the spectrum* Ω is not obvious from the algorithm, but can be derived from a dilation argument as follows:

If B is dilation-invariant, then it suffices to prove the sampling theorem for B^Q only, where $Q\subseteq R^n$ is the unit cube. Let R be the $n\times n$ diagonal matrix with entries $r_1, r_2, ..., r_n>0$ and consider the following (anisotropic) dilation of a function f

$$D_R f(x)=(\det R)\, f(R\cdot x).$$

All weighted L^p_w-spaces with a weight $w(x)=(1+|x|)^a$, $a\in R$, are invariant under D_R. Given $\Omega\subseteq R^n$ compact, choose a diagonal matrix $R=(r_1, r_2, ..., r_n)$ (with the r_i as large as possible), such that $R\cdot\Omega\subseteq Q$. Since $(D_R f)\hat{}\,(t)=\hat{f}(R^{-1}\cdot t)$, this means that $\operatorname{spec} D_R f\subseteq R\cdot\Omega\subseteq Q$ and $D_R f\in B^Q$. Now assume that a sampling set $X=(x_i)_{i\in I}$ is dense enough to allow a reconstruction of a function in B^Q as in Theorem 1. Then

$$D_R f(x)=\sum_{i\in I} D_R f(x_i) e_i(x)$$

or

$$f(R\cdot x)=\sum_{i\in I} f(R\cdot x_i) e_i(x) \tag{80}$$

$$f(x)=\sum_{i\in I} f(R\cdot x_i) e_i(R^{-1}\cdot x). \tag{81}$$

This transformation is admitted because B is D_R-invariant. In other words, $(R\cdot x_i)_{i\in I}$ is a sampling set for B^Ω. If, for instance, $\Omega=\prod_{i=1}^n[-r_i, r_i]$, then $R=(1/r_1, 1/r_2, ..., 1/r_n)$ and thus the sampling density in the ith coordinate is inversely proportional to the corresponding linear extension of the spec of f in this direction. This behavior is in complete agreement with the regular sampling theorem in several dimensions [36].

Remark 5.11. In order to obtain a series expansion of a band-limited function with respect to translates of another function g, the assumptions of Theorems 2–3 are also necessary.

(a) If $g \notin L^1_\alpha$, there is no reasonable way to express $f \in L^1_\alpha$ as $f = \sum c_i L_{x_i} g$ with a correct behavior of the coefficients $c_i \in l^1_\alpha$ and convergence of the sum in L^1_α. This is already the case for the classical cardinal series which looses its power outside L^2 and fails to converge in L^1.

(b) Given $g \in L^1_\alpha$ band-limited. Assume that every $f \in L^1_\alpha$ with $\operatorname{spec} f \subseteq \Omega$ has an expansion

$$f = \sum c_i L_{x_i} g \tag{82}$$

for some sequence $(x_i) \subseteq R^n$ and coefficients $(c_i) \in l^1_\alpha$. Then $\hat{g}(t) \neq 0$ on $\operatorname{int} \Omega$, the interior of Ω.

Proof. Assume on the contrary that $\hat{g}(t_0) = 0$ for some $t_0 \in \operatorname{int} \Omega$ and choose an $f \in L^1_\alpha$, $\operatorname{spec} f \subseteq \Omega$ and $\hat{f}(t_0) \neq 0$, e.g., a Schwartz function $\hat{f}$ with support in a small neighborhood of t_0 contained in Ω. Then the Fourier transfom of (82) is

$$\hat{f}(t) = \left(\sum_j c_j e^{i x_j t} \right) \hat{g}(t). \tag{83}$$

Since the sum in the bracket converges absolutely to an almost periodic function, we arrive at a contradiction for $t = t_0$.

(c) Given $g \in L^1_\alpha$ such that $\hat{g}(t) \neq 0$ everywhere. If a band-limited function $f \in L^1_\alpha$ has a representation (83), then $f \equiv 0$. Note that no further assumption on the x_i and $(c_i) \in l^1_\alpha$ is made except their existence.

Proof. Since $\hat{g}(t) \neq 0$ for all t, $\phi(t) = \sum_j c_j e^{i x_j t} \equiv 0$ outside $\operatorname{spec} f$. Now observe that ϕ is an almost periodic function and thus admits a continuous extension $\tilde{\phi}$ to the Bohr compactification bR^n [9, Chap. 15]. $\tilde{\phi}$ vanishes on $b(R^n \backslash \operatorname{spec} f)$, which is still a dense subset of bR^n, because $\operatorname{spec} f$ is compact, therefore $\tilde{\phi}$ must vanish identically. Hence $\phi \equiv 0$ and the conclusion follows.

6. Irregular Sampling in Locally Compact Abelian Groups

Theorems 1–3 remain true for arbitrary locally compact abelian groups G. Regular sampling in locally compact abelian groups, based on the existence of discrete, cocompact subgroups and Poisson's formula, has been considered in [31], special cases of it, known as Walsh sampling were treated by the Aachen shcool, see [6]. As far as we know, irregular sampling in locally compact abelian groups has not been investigated before. Using our approach of Sections 2–4, only minor modifications are needed to derive such theorems for general locally compact abelian groups.

We assume that G is a non-compact, locally compact abelian group. A distributional Fourier transform is then available on the space of distributions $S_0'(G)$, which is a good substitute for the tempered distributions on locally compact abelian groups. See [11, 12] for an exposition. Analogous to 2.1 we consider Banach spaces B of locally integrable functions, which are imbedded into $S_0'(G)$, translation invariant and Banach lattices, i.e., $|f| \leqslant |g|$ a.e. and $g \in B$ implies $f \in B$ and $\|f\|_B \leqslant \|g\|_B$. Moreover, a Beurling algebra $L^1_w(G)$ with w being a Beurling–Domar weight, see, e.g., [37] for the necessary background, is supposed to act on B by convolution: $L^1_w * B \subseteq B$. Then the Fourier transform on these spaces is well defined at least as a distribution in $S_0'(\hat{G})$, and one may consider the spaces of band-limited functions $B^\Omega = \{f \in B : \operatorname{supp} \hat{f} \subseteq \Omega\}$, where Ω is a compact subset in $\hat{G}$.

Since $L^1_w(G)$ is a Beurling algebra, there always exist band-limited functions $h \in L^1_w$ such that $\hat{h}(\xi) = 1$ for all $\xi \in \Omega$ and $\hat{h}(\xi) = 0$ outside some open, relatively compact neighborhood Ω_0 of Ω. Consequently a reproducing formula $f = f * h$ is available for all $f \in B^\Omega$ and the convolution operator $C_h f = f * h$ on B^{Ω_0} can be analyzed as in Section 3.

If no metric is available on G, then we have to redefine the density of a discrete set and the oscillation of a function in terms of a neighborhood U. A discrete set $X = (x_i)_{i \in I}$ is U-dense, if $\bigcup_{i \in I} (x_i + U) = G$. The U-oscillation $\operatorname{osc}_U f$ is defined by $\operatorname{osc}_U f(x) = \sup_{u \in U} |f(x+u) - f(x)|$. This more general type of oscillation could also be of use for metric groups or in R^n for a fine tuning of the required sampling density. Instead of a uniform density δ in all directions the density can be adjusted to the geometry of the spectrum Ω by using different shapes for U instead of the balls $B_\delta(0)$.

THEOREM 4. *Let G be a non-compact locally compact abelian group, B a Banach space of locally integrable functions with the properties stated above, $\Omega \subseteq \hat{G}$ a compact set, and $h \in L^1_w(G)$ a function such that $\hat{h}(t) = 1$ for $t \in \Omega$ and $\hat{h}(t) = 0$ outside an open neighborhood Ω_0 of Ω. Then there exists a neighborhood U of the identity such that any $f \in B^\Omega$ can be completely and stably reconstructed from any U-dense, discrete set $(x_i)_{i \in I}$ of G.*

The reconstruction can be carried out by the iteration procedure (53)–(55). *Alternatively there exist functions $e_i \in L^1_w(G)$, $\operatorname{supp} e_i \subseteq \Omega_0$ such that*

$$f = \sum_{i \in I} f(x_i) e_i$$

where the series converges in B and uniformly on compact sets.

Proof. Since the proof is almost the same as for Theorem 1, we indicate only the necesary modifications. If $\Psi = (\psi_i)_{i \in I}$ is some partition of unity associated to the U-dense set $(x_i)_{i \in I}$ in G, i.e., $0 \leqslant \psi_i \leqslant 1$, $\sum_{i \in I} \psi_i = 1$, $\operatorname{supp} \psi_i \subseteq x_i + U$, the approximation operator A_1 is defined as in (16). Then Lemma 3 is still valid, i.e., $\|C_h f - A_1 f\|_B \leqslant c(U, \Omega_0) \, \|h\|_{1,w} \, \|f\|_B$, where $c(U, \Omega_0) = \inf \|\operatorname{osc}_U p\|_{1,w}$, p ranging over all functions $p \in L^1_w(G)$ with $\hat{p}(t) = 1$ on Ω_0. It only has to be shown that $c(U, \Omega_0) \to 0$, as U shrinks to the identity $U \to \{e\}$: Given a band-limited $p \in L^1_w$, a reference neighborhood U_0, and $\varepsilon > 0$, one can find a compact set $K \subseteq G$, such that $\int_{G \setminus K} \operatorname{osc}_U p < \varepsilon$ for all $U \subseteq U_0$. Since p is uniformly continuous on G, there exists a neighborhood U of e, such that $\int_K \operatorname{osc}_U p < \varepsilon$ and consequently $c(U, \Omega_0) < 2\varepsilon$.

The proof of Proposition 6 and Theorem 1 is now literally the same. Analogs of Theorems 2 and 3 are proven similarly.

Remark. Let $H(\Omega)$ be the subgroup of $\hat{G}$ generated by the compact set Ω. Since for $f \in B^\Omega$ the support $\operatorname{supp} f \subseteq \Omega \subseteq H(\Omega)$, f has to be constant on the cosets of the annihilator $H(\Omega)^\perp = \{x \in G : \chi(x) = 1, \ \forall \chi \in H(\Omega)\}$. Therefore $f \in B^\Omega$ can be identified with a function on the quotient group $G/H(\Omega)^\perp \cong H(\Omega)^\wedge$. By the structure theorem, cf. [22], for compactly generated locally compact abelian groups $H(\Omega)$ is of the form $R^n \times Z^n \times F$ for some compact group F. Consequently $H(\Omega)^\wedge \cong R^n \times T^n \times D$, where T is the torus group and $D = \hat{F}$ is a discrete group. Thus from an abstract point of view only sampling on R^n is needed.

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